

① Theorem 7 :→

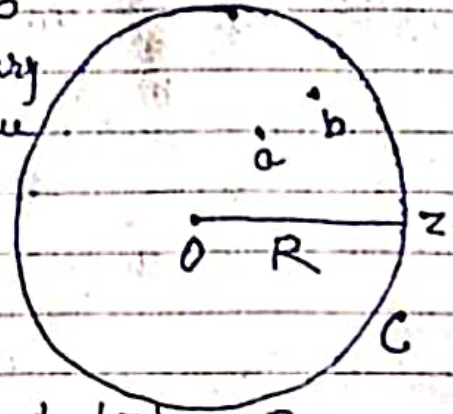
State and prove Liouville's theorem. 14 मंगल
Statement :→ If an entire function is bounded for all values of z , then it is constant.

or

If a function $f(z)$ is analytic for finite value of z , and is bounded then $f(z)$ is constant.

(2)

15 कुछ Proof: \rightarrow Let a and b be any two arbitrary distinct points in z -plane and let C be a large circle with centre $z=0$ and radius R so that it encloses a and b .



The equation of the circle is $|z| = R$

$$\therefore z = R e^{i\theta}, \quad dz = i R e^{i\theta} d\theta$$

$$|dz| = R d\theta$$

Since $f(z)$ is bounded for all C

$$\therefore |f(z)| \leq M \quad \forall z \text{ where } M > 0$$

By Cauchy's Integral formula

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$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z-a}, \quad f(b) = \frac{1}{2\pi i} \int_C \frac{f(z) dz}{z-b}$$

$$\therefore f(a) - f(b) = \frac{1}{2\pi i} \int_C \left(\frac{1}{z-a} - \frac{1}{z-b} \right) f(z) dz$$

$$= \frac{a-b}{2\pi i} \int_C \frac{f(z) dz}{(z-a)(z-b)}$$

$$\therefore |f(a) - f(b)| < \frac{|a-b|}{2\pi} \int_C \frac{|f(z)| |dz|}{(|z|-|a|)(|z|-|b|)}$$

?

$$\begin{aligned}
 \alpha |f(a) - f(b)| &\leq \frac{|a-b| \cdot M \cdot 2\pi R}{2\pi (R-|a|)(R-|b|)} \\
 &\leq M |a-b| \frac{1}{R} \\
 &\qquad \qquad \qquad \rightarrow 0 \\
 &\qquad \qquad \qquad \left(1 - \frac{|a|}{R}\right) \left(1 - \frac{|b|}{R}\right) \rightarrow 0
 \end{aligned}$$

$\therefore f(a) - f(b) = 0$ Or $f(a) = f(b)$
 which shows that $f(z)$ is constant
Proved